

ESTIMATING THE TRACE-FREE RICCI TENSOR IN RICCI FLOW

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ABSTRACT. An important and natural question in the analysis of Ricci flow singularity formation in all dimensions $n \geq 4$ is the following: What are the weakest conditions that provide control of the norm of the Riemann curvature tensor? In this short note, we show that the trace-free Ricci tensor is controlled in a precise fashion by the other components of the irreducible decomposition of the curvature tensor, for all compact solutions in all dimensions $n \geq 3$, without any hypotheses on the initial data.

1. INTRODUCTION

Standard short-time existence results imply that a solution $(\mathcal{M}^n, g(\cdot))$ of Ricci flow on a compact manifold becomes singular at $T < \infty$ if and only if

$$\lim_{t \nearrow T} \max_{x \in M^n} |\text{Rm}(x, t)| = \infty.$$

This suggests the following natural question: What are the weakest conditions that provide control of the norm of the full curvature tensor on a manifold evolving by Ricci flow?

In any dimension, it is true that a finite-time singularity happens if and only if

$$\limsup_{t \nearrow T} \max_{x \in M^n} |\text{Rc}(x, t)| = \infty.$$

Nataša Šešum has given a direct proof [4]. The claim also follows from independent results of Miles Simon [5] by a short argument.¹

In dimension three, an eminently satisfactory answer is given by the well known pinching theorem obtained independently by Ivey [3] and Hamilton [2]. Their estimate implies in particular that the scalar curvature dominates the full curvature tensor of any Ricci flow solution on a compact 3-manifold with normalized initial data.²

Xiuxiong Chen has expressed hope that an appropriate bound on the scalar curvature might be sufficient to rule out singularity formation in all dimensions.

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¹Let $(\mathcal{M}^n, g(t))$ be a solution of Ricci flow on a compact manifold such that $g(t)$ is smooth for $t \in [0, T)$, where $T < \infty$. If $\limsup_{t \nearrow T} (\max_{x \in \mathcal{M}^n} |\text{Rc}(x, t)|) < \infty$, then [1, Lemma 14.2] guarantees existence of a complete C^0 limit metric $g(T)$. One may then apply [5, Theorem 1.1], choosing a background metric $\bar{g} := g(T - \delta)$ such that $(1 - \varepsilon)\bar{g} \leq g \leq (1 + \varepsilon)\bar{g}$, where $\varepsilon = \varepsilon(n)$ and $\delta = \delta(\varepsilon)$. Let $\bar{K} = \max_{x \in \mathcal{M}^n} |\text{Rm}(\bar{g})|_{\bar{g}}$. Simon's theorem implies that there exists $\eta = \eta(n, \bar{K})$ such that for any $\theta \in [0, \delta]$, a solution $\hat{g}(s)$ of harmonic-map-coupled Ricci flow exists for $0 \leq s < \eta$ and satisfies $\hat{g}(0) = g(T - \theta)$; moreover, $\hat{g}(s)$ is smooth for $0 < s < \eta$. Since harmonic-map-coupled Ricci flow is equivalent to Ricci flow modulo diffeomorphisms, the claim follows by taking $\theta = \eta/2$.

²The Hamilton–Ivey pinching estimate implies the much stronger result that any rescaled limit of a finite time singularity in dimension three must have nonnegative sectional curvature.

Partial progress toward this conjecture was made recently by Bing Wang [6]. He proved that if the Ricci tensor is uniformly bounded from below on $[0, T)$ and if an integral bound

$$\int_0^T \int_{\mathcal{M}^n} |R|^\alpha d\mu dt < \infty$$

holds for some $\alpha \geq (n+2)/2$, then no singularity occurs at time $T < \infty$.

Recall that in any dimension $n \geq 3$, the Riemann curvature tensor admits an orthogonal decomposition

$$\text{Rm} = U + V + W$$

into irreducible components

$$U = \frac{1}{2n(n-1)} R(g \bar{\wedge} g), \quad V = \frac{1}{n-2} (F \bar{\wedge} g), \quad W = \text{Weyl tensor},$$

where $\bar{\wedge}$ denotes the Kulkarni–Nomizu product of symmetric tensors and F denotes the trace-free Ricci tensor. The purpose of this short note is to observe that V is always dominated by the other components in the following sense:

Main Theorem. *If $(\mathcal{M}^n, g(\cdot))$ is a solution of Ricci flow on a compact manifold of dimension $n \geq 3$, then there exist constants $c(g_0) \geq 0$, $C_1(n, g_0) > 0$, and $C_2(n) > 0$ such that for all $t \geq 0$ that a solution exists, one has $R + c > 0$ and*

$$\frac{|V|}{R+c} \leq C_1 + C_2 \max_{s \in [0, t]} \sqrt{\frac{|W|_{\max}(s)}{R_{\min}(s) + c}}.$$

2. PROOF OF THE MAIN THEOREM

Define

$$a = |F| = \frac{\sqrt{n-2}}{2} |V|,$$

noting that a is smooth wherever it is strictly positive. Choose $c \geq 0$ large enough so that $R_{\min}(0) + c > 0$ and define

$$b = R + c,$$

noting that $b > 0$ for as long as a solution exists.

In any dimension $n \geq 3$, one has

$$\frac{\partial}{\partial t} |F|^2 = \Delta |F|^2 - 2 |\nabla F|^2 + \frac{4(n-2)}{n(n-1)} R |F|^2 - \frac{8}{n-2} \text{tr } F^3 + 4W(F, F),$$

where $\text{tr } F^3 = F_i^j F_j^k F_k^i$ and $W(F, F) = W_{ijkl} F^{il} F^{jk}$. It follows from Cauchy–Schwarz that a obeys the differential inequality

$$a_t \leq \Delta a + \frac{2(n-2)}{n(n-1)} a(b-c) - \frac{4}{n-2} a^{-1} \text{tr } F^3 + 2a^{-1} W(F, F).$$

The positive quantity b evolves by

$$b_t = \Delta b + 2a^2 + \frac{2}{n} (b-c)^2.$$

To prove the theorem, it will suffice to bound the scale-invariant non-negative quantity

$$\varphi = \frac{a}{b}.$$

Because $\Delta\varphi = b^{-1}(\Delta a - \varphi\Delta b) - 2\langle\nabla\varphi, \nabla\log b\rangle$, one has

$$\varphi_t \leq \Delta\varphi + 2\langle\nabla\varphi, \nabla\log b\rangle + 2\rho\varphi,$$

where the reaction term is

$$\rho = \frac{n-2}{n(n-1)}(b-c) - \frac{2}{n-2}\frac{\operatorname{tr} F^3}{a^2} + \frac{W(F, F)}{a^2} - \frac{(b-c)^2}{nb} - a\varphi.$$

There exist positive constants c_1, c_2 depending only on $n \geq 3$ such that

$$\left| \frac{2}{n-2} \operatorname{tr} F^3 \right| \leq c_1 a^3 \quad \text{and} \quad |W(F, F)| \leq c_2 |W| a^2.$$

Hence

$$\rho \leq \frac{n-2}{n(n-1)}(b-c) + c_1 a + c_2 |W| - \frac{(b-c)^2}{nb} - a\varphi.$$

Define constants α, β, γ by $\alpha^2 = c_1$, $\beta^2 = \frac{n-2}{n(n-1)}$, and $\gamma^2 = c_2$. Fix $\varepsilon > 0$ and choose $C_1 = \max\{\alpha^2 + \beta, \varphi_{\max}(0) + \varepsilon\}$ and $C_2 = \gamma$. Consider the barrier function

$$\Phi(t) = C_1 + C_2 \max_{s \in [0, t]} \sqrt{\frac{|W|_{\max}(s)}{b_{\min}(s)}},$$

noting that Φ is monotone nondecreasing. If $\varphi_{\max}(t) \geq \Phi(t)$ at some $t > 0$, then at any (x, t) where φ attains its spatial maximum, one has

$$a \geq (\alpha^2 + \beta)b + \gamma\sqrt{b|W|},$$

which implies that

$$\begin{aligned} a^2 &\geq \alpha^2 ab + \beta^2 b^2 + \gamma^2 b |W| \\ &\geq \frac{n-2}{n(n-1)}(b^2 - bc) + (c_1 a + c_2 |W|) b - \frac{1}{n}(b-c)^2, \end{aligned}$$

hence that $\rho \leq 0$, hence that $\varphi_t \leq 0$, hence that $\frac{d^+}{dt}\varphi_{\max}(t) \leq 0$, understood in the usual sense as the lim sup of difference quotients. It follows that $\varphi_{\max}(t) \leq \Phi(t)$ for as long as a solution exists.

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REFERENCES

- [1] **Hamilton, Richard S.** Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** (1982), no. 2, 255–306.
- [2] **Hamilton, Richard S.** The formation of singularities in the Ricci flow. *Surveys in differential geometry, Vol. II* (Cambridge, MA, 1993), 7–136, Internat. Press, Cambridge, MA, 1995.
- [3] **Ivey, Thomas.** Ricci solitons on compact three-manifolds. *Differential Geom. Appl.* **3** (1993), no. 4, 301–307.
- [4] **Šešum, Nataša.** Curvature tensor under the Ricci flow. *Amer. J. Math.* **127** (2005), no. 6, 1315–1324.
- [5] **Simon, Miles.** Deformation C^0 Riemannian metrics in the direction of their Ricci curvature. *Comm. Anal. Geom.* **10** (2002), no. 5, 1033–1074.
- [6] **Wang, Bing.** On the conditions to extend Ricci flow. [arXiv:math.DG/0704.3018](https://arxiv.org/abs/math/0704.3018).

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